1. Production Possibility Frontier

We now move from partial equilibrium, where we examined only one market at a time, to general equilibrium, where we examine multiple markets interactions\(^1\). We start with the most simple example where there is one input, labor \((L)\), and two outputs, guns \((X)\), and butter \((Y)\). For each good, \(X\) and \(Y\), we have a separate production function. Let’s define these as,

\[
X = \sqrt{L_X} \quad Y = \frac{1}{2} \sqrt{L_Y}
\]

In contrast to the analysis we have done before, we now say that the input \(L\) is limited, so that an increase in \(L\) used to make \(X\), must decrease the amount of \(L\) to make \(Y\). This is why we write \(L_X\) and \(L_Y\), to denote where the labor is used. For this example, we assume the maximum labor available is 100. Therefore the resource constraint on labor is,

\[
L_X + L_Y = 100
\]

First note that if all the labor was used to make \(X\), we would have,

\[
X = \sqrt{100} = 10
\]

Similarly, if all the labor was used to make \(Y\),

\[
Y = \frac{1}{2} \sqrt{100} = 5
\]

Next, we wish to solve for the production possibility frontier. This describes the relation between producing the two goods, \(X\) and \(Y\). We can rewrite our production functions as follows,

\[
X = \sqrt{L_X} \\
X^2 = L_X
\]

\(^1\)This section parallels Example 16.1 from Nicholson.
With these two equations for $L_X$ and $L_Y$, we substitute into the resource constraint for labor,

\[ L_X + L_Y = 100 \]

\[ \implies \text{PPF} \equiv X^2 + 4Y^2 = 100 \]

Notice that the PPF is an implicit function of $X$ and $Y$, so we can solve for the slope $\frac{dy}{dx}$ using the techniques of Lecture Notes 1, Section 2. Recall,

\[ F(X,Y) = X^2 + 4Y^2 - 100 = 0 \]

\[ \frac{dF(X,Y)}{dX} = 2X + 8Y \cdot \frac{dY}{dX} = 0 \]

\[ \implies \frac{dY}{dX} = \frac{-2X}{8Y} \]

\[ \frac{dY}{dX} = \frac{-X}{4Y} \]

What is this slope? It is the negative of the Rate of Product Transformation (RPT). What is the slope if only $X$ is produced? Remember we found that if all the labor went to make $X$, the total amount of $X$ produced is 10, and $Y = 0$.

\[ RPT(X = 10, Y = 0) = \frac{2 \cdot 10}{0} = \infty \]

What about if only $Y$ is produced?

\[ RPT(X = 0, Y = 5) = \frac{0}{20} = 0 \]

So let’s graph the PPF. See Figure 1. Notice that the slope is zero at (0,5) and infinite at (10,0). Therefore, we know that the graph must be bowed outward as shown.

2. General Equilibrium Pricing

Now we wish to find the prices that will obtain in the markets for $X$ and $Y$. Just as supply and demand in partial equilibrium determined the price, here we have a similar concept. Demand is represented by a utility function and supply is represented by the PPF. The utility function gives a relation between the two goods $X$ and $Y$, just as the PPF does. For example, take the following utility,

\[ \text{utility} = U(X,Y) = \sqrt{XY} = X^{\frac{1}{2}}Y^{\frac{1}{2}} \]

Notice that this is the same function we graphed in TA Lecture 2. The consumer can choose any $X$ and $Y$ that can be produced with $L \leq 100$. This is the shaded region of Figure 1. Naturally, to maximize utility, the consumers will want their indifference curve to be just tangent to the PPF. See Figure 2. Notice that the points along $U_1$ in the shaded region are all feasible allocations of $X$ and $Y$, but the consumer can do better at $U_2$. Here there is just one point of tangency and this
allocation of $X$ and $Y$ is optimal. The points on $U_3$ are preferred to all the points on $U_2$, but they are not feasible allocations. So how do we solve for the one point of tangency of PPF and $U_2$? We know that at any tangency point, the slopes of the two curves must be equal. Therefore, we can set the MRS equal to the RPT. We have solved for the RPT, but what is the MRS?

$$MRS = \frac{MU_X}{MU_Y} = \frac{Y}{X}$$

Therefore to find the allocation $X, Y$ at the tangency point,

$$MRS = RPT$$

$$Y = X$$

$$\frac{X}{4Y}$$

$$4Y^2 = X^2$$

We solve for the particular $X$ and $Y$ that are on the PPF, so we have,

$$PPF \equiv X^2 + 4Y^2 = 100$$

$$X^2 + X^2 = 100$$

$$2X^2 = 100$$

$$X^2 = 50$$

$$X = 5\sqrt{2} = 7.07$$
We substitute this value back into the above equation for $X$,

$$
50 + 4Y^2 = 100 \\
4Y^2 = 50 \\
Y^2 = \frac{50}{4} \\
Y = \frac{5\sqrt{2}}{2} = 3.54.
$$

Therefore, the optimal allocation to the consumers is $(X^*, Y^*) = (7.07, 3.54)$. Just as in the partial equilibrium analysis, we found the quantity where demand equaled supply. At this point, we also found the equilibrium price. What is the equilibrium price in the general equilibrium case?

The prices in the GE case are found as a ratio of one to the other, $\frac{P_X}{P_Y}$, and this ratio is simply the slope of the utility indifference curve at the point of tangency with the PPF. Therefore we have the following,

$$RPT = \frac{P_X}{P_Y} = MRS$$
We have already computed this slope above, so we have at the optimal allocation

\[
\frac{P_X}{P_Y} = \frac{Y}{X} = MRS
\]

\[
= \frac{3.54}{\frac{7.07}{2}}
\]

See Figure 3 for these relations.

**Figure 3.** Optimal Allocation and Prices

### 3. Comparative Statics

Making the connection between the partial equilibrium model and the general equilibrium model, we ask how does the equilibrium change if demand changes. In partial equilibrium, we see the demand curve expand or contract and the prices and quantities adjust to find a new equilibrium. In the GE context, the same fundamentals apply, but now if demand for X increases and so the quantity of X supplied increases, labor is taken away from the production of good Y. Using the same example as above, now assume that the utility shifts to favor good X,

\[
U(X, Y) = X^{\frac{3}{4}} Y^{\frac{1}{4}}
\]

To solve for the new equilibrium allocation of X and Y, as well as the new price ratio, we find the

\[
MRS = \frac{\partial U}{\partial X} = \frac{3Y}{X}
\]
Production functions have not changed, so the PPF has not changed, and so the RPT has not changed. Therefore, the new equilibrium allocation is solved as before,

\[ MRS = RPT \]
\[ \frac{3Y}{X} = \frac{X}{4Y} \]
\[ 12Y^2 = X^2 \]

Substituting into the equation for the PPF,
\[ X^2 + 4Y^2 = 12Y^2 + 4Y^2 = 100 \]
\[ = 16Y^2 = 100 \]
\[ = Y^2 = \frac{100}{16} \]
\[ = \frac{Y^2}{4} = \frac{25}{4} \]
\[ = Y^* = \frac{5}{2} = 2.5 \]

Now find \( X^* \),
\[ 12Y^2 = X^2 \]
\[ 12 \cdot \frac{25}{4} = X^2 \]
\[ 75 = X^2 \]
\[ 5\sqrt{3} = X^* = 8.66 \]

Let’s find the price ratio,
\[ \frac{P_X}{P_Y} = \frac{3Y}{X} = MRS \]
\[ = \frac{3 \cdot 2.5}{8.66} \]
\[ = \frac{7.5}{8.66} \]
\[ = \frac{0.866}{1} \]

See Figure 4 for this new equilibrium. Notice that the optimal allocation of \( X \) increased and \( Y \) decreased, and the relative price of \( X \) has also increased. This makes sense given the new utility function which has a higher preference for \( X \).
Figure 4. A Change in Preferences